

Coulomb Scattering of s^+

This is the calculation of the transition amplitude for a scalar charged particle s^+ being scattered by a classical coulomb potential. The approach is field-theoretic.

As the s^+ particle is being scattered by a Coulomb potential, the appropriate interaction Hamiltonian for use in the Dyson Series

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \dots \int d^4x_1 \dots d^4x_n T[\mathcal{H}(x_1) \dots \mathcal{H}(x_n)]$$

is

$$\mathcal{H}_{s^+} = -\mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = -iq \underbrace{(\phi^\dagger \cdot \partial^\mu \phi - \partial^\mu \phi \cdot \phi)}_{\text{1st order EM interaction}} A_\mu + q^2 \underbrace{A^\mu A_\mu \phi^\dagger \phi}_{\text{2nd order EM interaction}}$$

where the charge of the particle is $q=e$.

Consider the interaction to 1st order perturbation; ignore the q^2 term.

$$\mathcal{H}'_{s^+} = ie (\phi^\dagger (\partial^\mu \phi) - (\partial^\mu \phi) \phi) A_\mu$$

So the 1st order transition matrix element (or "probability amplitude") is

$$S_{fi}^{(1)} = \langle s^+(p') | S^{(1)} | s^+(p) \rangle$$

$$= \langle s^+(p') | \int dt (-iH'(t)) | s^+(p) \rangle$$

where $H'(t)$ is the interaction term of the Hamiltonian in the interaction picture, $H' = \int d^3x \lambda'$.

$$S_{fi}^{(1)} = \langle s^+(p') | \int dt d^3x (-i) \lambda' | s^+(p) \rangle$$

$$= -i \langle s^+(p') | \int d^4x \lambda' | s^+(p) \rangle$$

$$= -i \langle s^+(p') | \int d^4x [ie(\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi)\phi)] | s^+(p) \rangle A_\mu$$

$$= -i \int d^4x \langle s^+(p') | [ie(\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi)\phi)] | s^+(p) \rangle A_\mu$$

Define $j_{ext}^\mu(x) \equiv ie(\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi)\phi)$

$$S_{fi}^{(1)} = -i \int d^4x \langle s^+(p') | j_{ext}^\mu(x) | s^+(p) \rangle A_\mu$$

Now the objective is to calculate the 4-current matrix element that appears in the integrand.

A few precursors first: The normalization condition for the states is

$$|s^+(\vec{p})\rangle = \sqrt{2E} a^+(\vec{p}) |0\rangle$$

and the field expansions are

$$\phi = \int \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2\omega}} [a(\vec{k}) e^{-ik \cdot x} + b^\dagger(\vec{k}) e^{ik \cdot x}]$$

$$\partial^\mu \phi = \int \frac{d^3 \vec{k} (-ik^\mu)}{(2\pi)^3 \sqrt{2\omega}} [a(\vec{k}) e^{-ik \cdot x} - b^\dagger(\vec{k}) e^{ik \cdot x}]$$

Using these, the 4-current amplitude becomes

$$\langle s^+(\vec{p}') | j_\mu^P(x) | s(\vec{p}) \rangle$$

$$= \langle 0 | \sqrt{2E'} a(\vec{p}') \cdot i e [(\phi^\dagger \partial^\mu \phi) - (\partial^\mu \phi) \phi] \sqrt{2E} a^+(\vec{p}) | 0 \rangle$$

$$= i e \sqrt{4E'E} \left[\underbrace{\langle 0 | a(\vec{p}') \phi^\dagger (\partial^\mu \phi) a^+(\vec{p}) | 0 \rangle}_{(I)} - \underbrace{\langle 0 | a(\vec{p}') (\partial^\mu \phi) \phi a^+(\vec{p}) | 0 \rangle}_{(II)} \right]$$

Evaluate (I):

$$\langle 0 | a(\vec{p}') \int \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2\omega}} [a^\dagger(\vec{k}) e^{ik \cdot x} + b(\vec{k}) e^{-ik \cdot x}]$$

$$\cdot \int \frac{d^3 \vec{k}' (-ik'^\mu)}{(2\pi)^3 \sqrt{2\omega'}} [a(\vec{k}') e^{-ik' \cdot x} - b^\dagger(\vec{k}') e^{ik' \cdot x}] a^+(\vec{p}) | 0 \rangle$$

$$= \langle 0 | \left\{ \left(\frac{d^3 k}{(2\pi)^6 \sqrt{2\omega} \sqrt{2\omega'}} \right) \alpha(p') \left[a^\dagger(k) e^{ik \cdot x} + b^\dagger(k) e^{-ik \cdot x} \right] \right. \\ \left. \cdot \left[a(k') e^{-ik' \cdot x} - b^\dagger(k') e^{ik' \cdot x} \right] a^\dagger(p) | 0 \rangle \right)$$

$$= \langle 0 | \left\{ \left(\frac{d^3 k}{(2\pi)^6 \sqrt{2\omega} \sqrt{2\omega'}} \right) \alpha(p') \left[a^\dagger(k) a(k') e^{i(k-k') \cdot x} + b^\dagger(k) a(k') e^{-i(k+k') \cdot x} \right. \right. \\ \left. \left. - a^\dagger(k) b^\dagger(k') e^{i(k+k') \cdot x} - b^\dagger(k) b^\dagger(k') e^{-i(k-k') \cdot x} \right] a^\dagger(p) | 0 \rangle \right)$$

$$= \langle 0 | \left\{ \left(\frac{d^3 k}{(2\pi)^6 \sqrt{2\omega} \sqrt{2\omega'}} \right) \left[\alpha(p') a^\dagger(k) a(k') a^\dagger(p) e^{i(k-k') \cdot x} \right. \right. \\ \left. + \alpha(p') b^\dagger(k) a(k') a^\dagger(p) e^{-i(k+k') \cdot x} \right. \\ \left. - \alpha(p') a^\dagger(k) b^\dagger(k') a^\dagger(p) e^{i(k+k') \cdot x} \right. \\ \left. \left. - \alpha(p') b^\dagger(k) b^\dagger(k') a^\dagger(p) e^{-i(k-k') \cdot x} \right] | 0 \rangle \right)$$

Recall that the a -operators commute with the b -operators. Then the middle two terms cancel b/c $b^\dagger(k) b(k) | 0 \rangle = 0$ and $\langle 0 | b^\dagger(k) = 0$. I'm not completely sure why the 4th term cancels, but it seems the a -operators produce the s^+ particle and the b -operators produce something else b/c it commutes w/ the a -operators. This interaction ends w/ a single $s^+(p')$ and begins w/ a single $s^-(p)$. So that something else from the b -operator does not belong.

$$\Rightarrow \langle 0 | \left\{ \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2\omega}} a(\vec{p}) a^\dagger(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \right\} \left\{ \frac{d^3 \vec{k}'}{(2\pi)^3 \sqrt{2\omega'}} a(\vec{k}') a^\dagger(\vec{p}) e^{-i\vec{k}' \cdot \vec{x}} \right\} | 0 \rangle$$

Use the commutation relation

$$[a(\vec{k}'), a^\dagger(\vec{p})] = (2\pi)^3 \delta^3(\vec{k}' - \vec{p})$$

$$a(\vec{k}') a^\dagger(\vec{p}) = (2\pi)^3 \delta^3(\vec{k}' - \vec{p}) + a^\dagger(\vec{p}) a(\vec{k}')$$

in the 2nd integral.

$$\Rightarrow \langle 0 | \left\{ \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2\omega}} a(\vec{p}) a^\dagger(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \right\} \left[\frac{d^3 \vec{k}'}{(2\pi)^3 \sqrt{2\omega'}} \left[(2\pi)^3 \delta^3(\vec{k}' - \vec{p}) + a^\dagger(\vec{p}) a(\vec{k}') \right] \right] e^{-i\vec{k}' \cdot \vec{x}} | 0 \rangle$$

$$= \langle 0 | \left\{ \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2\omega}} a(\vec{p}) a^\dagger(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \right\} \left\{ \frac{d^3 \vec{k}'}{(2\pi)^3 \sqrt{2\omega'}} \delta^3(\vec{k}' - \vec{p}) e^{-i\vec{k}' \cdot \vec{x}} \right\} | 0 \rangle$$

$$= \langle 0 | \left\{ \frac{d^3 \vec{k} (-i\vec{p}')} {(2\pi)^3 \sqrt{4\omega\omega'}} a(\vec{p}) a^\dagger(\vec{k}) e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{p}' \cdot \vec{x}} \right\} | 0 \rangle$$

$$= \langle 0 | \left\{ \frac{d^3 \vec{k} (-i\vec{p}')} {(2\pi)^3 \sqrt{4\omega\omega'}} a(\vec{p}) a^\dagger(\vec{k}) e^{-i(\vec{p}-\vec{k}) \cdot \vec{x}} \right\} | 0 \rangle$$

Use the commutation relation

$$[a(\vec{p}'), a^\dagger(\vec{k})] = (2\pi)^3 \delta^3(\vec{p}' - \vec{k})$$

$$a(\vec{p}') a^\dagger(\vec{k}) = (2\pi)^3 \delta^3(\vec{p}' - \vec{k}) + a^\dagger(\vec{k}) a(\vec{p}')$$

Note also that b/c of the δ -functions

$$\begin{aligned} \omega &= E \\ \omega' &= E' \end{aligned}$$

$$\begin{aligned}
 &= \langle 0 | \int \frac{d^3 k (-i p'^*)}{(2\pi)^3 \sqrt{4 E E'}} [(2\pi)^3 \delta^3 (\vec{p}' - \vec{k}) + a^\dagger(\vec{k}) a(\vec{p}')] e^{-i(p-k) \cdot x} | 0 \rangle \\
 &= \langle 0 | \int \frac{d^3 k (-i p'^*)}{\sqrt{4 E E'}} \delta^3 (\vec{p}' - \vec{k}) e^{-i(p-k) \cdot x} | 0 \rangle \\
 &= \langle 0 | \frac{-i p'^*}{\sqrt{4 E E'}} e^{-i(p-p') \cdot x} | 0 \rangle \\
 &= \frac{-i p'^*}{\sqrt{4 E E'}} e^{-i(p-p') \cdot x}
 \end{aligned}$$

A nearly identical argument for (II) yields

$$\frac{-i p'^*}{\sqrt{4 E E'}} e^{-i(p-p') \cdot x}$$

Putting the pieces back together, we now have the transition amplitude.

$$\langle s^+(\vec{p}') | j_{s+}^P(x) | s^+(\vec{p}) \rangle = e^{(p'^* + p^*)} e^{-i(p-p') \cdot x}$$

$$S_{fi}^{(1)} = -i \int d^4 x \langle s^+(\vec{p}') | j_{s+}^P(x) | s^+(\vec{p}) \rangle A_p(x)$$